A General Proof of Floquet's Theorem

The purpose of this correspondence is to demonstrate the form of modes which are propagating in periodic structures. Although this form has long been used through an extension of Floquet's theorem for ordinary differential equations with periodic coefficients to modes propagating in periodic electromagnetic structures, (See [1]-[6]) its general validity for these modes, which are determined by partial differential equations, has not been demonstrated.

We shall employ an abstract notation similar to that used by Friedman [7] and Rumsey [8]. By a propagating mode we understand a source free solution to Maxwell's equations satisfying certain of the boundary conditions. For closed structures the transverse boundary conditions shall be satisfied, e.g., the waveguide modes in a waveguide. For open structures the boundary conditions on the structure shall be satisfied and the field shall decay expotentially away from the structure, e.g., surface waves on a dielectric slab. Thus, in abstract notation, if $\mathcal{S}$ is the space of all fields which satisfy the required boundary conditions, then Ψ is a mode if Ψ is in $\mathcal{S}$ and if Ψ satisfies the equation (Maxwell's equations)

$$L\Psi = 0 \tag{1}$$

i.e., the null space of L is the set of modes of the structure. For clarification, (1) expressed in matrix form is

$$\begin{pmatrix} -j\omega\epsilon & \nabla\times \\ \nabla\times & j\omega\mu \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix} = 0. \tag{2}$$

Let us suppose that z is the longitudinal coordinate along which the structure is periodic. We define the translation operator T by the following equation

$$T\Theta(z) = \Theta(z + p) \tag{3}$$

where p is the period of the structure. Since the structure is periodic, and since $\partial/\partial z = \partial/\partial(z+p)$, it is clear that the null space of L is an invariant subspace of T. In addition, we define the scalar product in $\mathcal{S}$ as follows (* denotes complex conjugate)

$$\langle\Psi, \Phi\rangle = \int_{-\infty}^{\infty} dz \iint_{\text{cross section}} dxdy\Psi^*(xyz)\Phi(x\,y, x). \tag{4}$$

The adjoint operator $\tilde{T}$ of T can be shown to be the inverse T^{-1} of T as follows

$$\begin{aligned}\langle\Psi, T\Phi\rangle &= \int_{-\infty}^{\infty} dz \iint_{\text{c.s.}} dxdy\Psi^*(x, y, z)\Phi(x, y, z + p) \\ &= \int_{-\infty}^{\infty} dz' \iint_{\text{c.s.}} dxdy \\ &\quad \Psi^*(x, y, z' - p)\Phi(x, y, z') \\ &= \langle T^{-1}\Psi, \Phi\rangle = \langle\tilde{T}\Psi, \Phi\rangle. \end{aligned} \tag{5}$$

Since T is not self-adjoint we can not assume that its eigenvectors span the null space $\mathcal{S}$ of L (modes of the structure). However,[1] we can assume that the generalized eigenvectors of T span the space $\mathcal{S}$. The generalized eigenvectors of T in $\mathcal{S}$ are all those vectors Ψ such that there exists an integer n and a scalar t such that

$$(T - t)^n\Psi = 0. \tag{6}$$

The rank of the generalized eigenvector is the smallest integer n such that (6) still holds. Thus, a generalized eigenvector of rank one is simply an eigenvector. We shall demonstrate that the operator T has no generalized eigenvectors of rank higher than one and thus its eigenvectors span the null space of L. (The null space of L is an invariant subspace of T.)

Suppose the Ψ' is a generalized eigenvector of T of rank two; then

$$(T - t)^2\Psi' = 0 \tag{7}$$

and

$$(T - t)\Psi' = \Phi \neq 0 \tag{8}$$

where Φ must be an eigenvector of T by virtue of (7). One notices that if $a\cdot\Phi$ is added to Ψ' that the resulting vector Ψ still satisfies (7) and (8), and is thus still a generalized eigenvector of rank two (where a is any constant). Also since $\langle\Phi, \Phi\rangle$ is a real constant greater than zero we can choose a to be

$$a = \frac{-\langle\Psi', \Phi\rangle}{\langle\Phi, \Phi\rangle} \tag{9}$$

and thus insure that

$$\langle\Psi, \Phi\rangle = \langle\Psi' + a\Phi, \Phi\rangle = \langle\Psi', \Phi\rangle - \frac{\langle\Psi', \Phi\rangle}{\langle\Phi, \Phi\rangle}\cdot\langle\Phi, \Phi\rangle = 0. \tag{10}$$

We now examine the following scalar product in light of (8) and (10).

$$\langle\Phi, (T - t)\Psi\rangle = \langle\Phi, \Phi\rangle \neq 0$$

but

$$\begin{aligned}\langle\Phi, (T - t)\Psi\rangle &= \langle\Phi, T\Psi\rangle - t\langle\Phi, \Psi\rangle = \langle\Phi, T\Psi\rangle \\ &= \langle T^{-1}\Phi, \Psi\rangle = \frac{1}{t^*}\langle\Phi, \Psi\rangle \\ &= 0. \end{aligned} \tag{11}$$

That t is nonzero is an obvious consequence of the fact that T is the translation operator.

Thus the assumption that T has a generalized eigenvector of rank two results in the contradiction shown in (11). Obviously, since T has no generalized eigenvectors of rank two, it can have no generalized eigenvectors of higher rank; for, if Ψ_m is a generalized eigenvector of rank $m > 2$ then

$$(T - t)^{m-2}\Psi_m = \Psi_2 \neq 0$$

and

$$(T - t)\Psi_2 \neq 0 \quad \text{but } (T - t)^2\Psi_2 = 0.$$

Hence, Ψ_2 is a generalized eigenvector of rank two which we have shown that T does not possess.

Since the generalized eigenvectors of T span the null space of L, and since all the generalized eigenvectors of T are ordinary eigenvectors, then the eigenvectors of T span the null space of L. Thus we may use the eigenvectors of T as a set of basis vectors for the modes of the periodic structure. Hence we can obtain all modes by vectors which satisfy the following relation

$$T\Phi = t\Phi. \tag{12}$$

Choose β_0 such that $e^{-j\beta_0 p} = t$, then from (12):

$$\Phi(z + p) = e^{-j\beta_0 p}\Phi(z). \tag{13}$$

Let

$$\Phi'(z) = e^{-j\beta_0 z}\Phi$$

then (13) becomes

$$e^{-j\beta_0(z+p)}\Phi'(z + p) = e^{-j\beta_0(z+p)}\Phi'(z)$$

or

$$\Phi'(z + p) = \Phi'(z)$$

Q.E.D.

For further elaboration, since Φ' is a periodic function of z, we may express it as a Fourier series in z,

$$\Phi'(x, y, z) = \sum_{-\infty}^{\infty} \Phi_n(x, y)e^{(-j2\pi n/p)z}.$$

Thus we have demonstrated that the form of modes propagating in periodic structures is the well known Floquet series:

$$\Phi(x, y, z) = \begin{bmatrix} E_x(x, y, z) \\ E_y(x, y, z) \\ E_z(x, y, z) \\ H_x(x, y, z) \\ H_y(x, y, z) \\ H_z(x, y, z) \end{bmatrix} = \sum_{-\infty}^{\infty} \begin{bmatrix} E_{xn}(x, y) \\ E_{yn}(x, y) \\ E_{zn}(x, y) \\ H_{xn}(x, y) \\ H_{yn}(x, y) \\ H_{zn}(x, y) \end{bmatrix} e^{-j(\beta_0 + 2n\pi/p)z}. \tag{14}$$

Similar relations can be derived in an analogous manner for structures with symmetries such that the null space of L is an invariant subspace of the particular symmetry operator involved and if the adjoint of the symmetry operator is the inverse of that symmetry operator.

For convenience, the function $F(r, \theta, z)$ is introduced to describe the various structure symmetry properties. For example, the geometry of periodic structures is described by

$$F(r, \theta, z + p) = F(r, \theta, z).$$

Other structures to which the techniques of this appendix can be applied are:

1) axial reflection symmetry (about the plane $z = z_0$)

$$F(r, \theta, 2z_0 - z) = F(r, \theta, z).$$

2) Angular rotation symmetry (about the structural axis)

$$F(r, \theta + \psi, z) = F(r, \theta, z).$$

3) Angular reflection symmetry (about the half plane $\theta = \theta_0$)

$$F(r, 2\theta_0 - \theta, z) = F(r, \theta, z).$$

4) Combined angular reflection symmetry (about the half plane $\theta = \theta_0$)

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[1] As in Friedman [7], p 111.

and axial reflection symmetry about the plane $z=z_0$)

$$F(r, 2\theta_0 - \theta, 2z_0 - z) = F(r, \theta, z).$$

5) Combined angular rotation symmetry (about the structural axis) and axial reflection symmetry (about the plane $z=z_0$)

$$F(r, \theta + \psi, 2z_0 - z) = F(r, \theta, z).$$

6) Screw symmetry (rotation about the structural axis combined with translation along it)

$$F(r, \theta + \psi, z + \delta) = F(r, \theta, z).$$

7) Glide symmetry (angular reflection about the half plane $\theta=\theta_0$ combined with translation of one-half period along the structural axis)

$$F(r, 2\theta_0 - \theta, z + p/2) = F(r, \theta, z).$$

Some important consequences of these symmetries are discussed in the article by Crepeau and McIsaac [9].

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Field Measurements in a Small Cross Section Guide Loaded with Magnetized Ferrite

By solving the Maxwell's equations for a rectangular guide partially filled with magnetized ferrite one finds that in particular conditions to be specified later for the geometry and the applied magnetic field only modes with phase velocity directed in one sense can propagate. This is the basis for the so-called thermodynamical paradox. The present work was carried out in order to experimentally investigate the microwave e.m. field in such a structure and compare the results with the theoretical predictions which we summarize here briefly.

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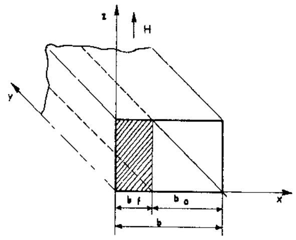

Fig. 1. Reference system.

We assume the configuration and notations shown in Fig. 1 (indefinite in the y direction). We assume for the microwave field a dependence $\exp(-k_x x - +k_y y - k_z z)$ with the coordinates.

The characteristic equation of the system was numerically solved by Barzilai and Gerosa for a number of cases ([1], [2], [3]) and the following main results were established:

1) For a given ferrite the solutions of the characteristic equation depend on the ratio ω/ω_0 between the working frequency and the resonant frequency $\omega_0=\gamma H$.

2) For $0<\mu_1<_{m2}$ (i.e., for

$$\sqrt{1 - \frac{\omega_m}{\omega_0}} < \frac{\omega}{\omega_0} < 1 + \frac{\omega_m}{\omega_0},$$

assuming the tensor permeability in the form:

$$\mu = \mu_0 \begin{vmatrix} \mu_1 & j\mu_2 & 0 \\ -j\mu_2 & \mu_1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

with

$$\mu_1 = 1 + \frac{\omega_0\omega_m}{\omega_0^2 - \omega^2} \quad \text{and} \quad \mu_2 = \frac{\omega\omega_m}{\omega_0^2 - \omega_m^2}$$

where $\omega_m=4\pi\gamma\mu_0$), a class of unidirectional propagating modes is found (the ferrite-dielectric and ferrite-metal modes of Seidel and Fletcher [6]). It is possible to choose the dimensions of the guide and the thickness of the ferrite slab in such a way that, within a given range of the applied magnetic field, these unidirectional modes be the only propagating modes, all higher order modes (the ferrite-guided modes following Lax's classification [5][1]) being under cutoff. The situation previously described gives rise to the so-called thermodynamical paradox, because all the propagating modes (nonattenuated for lossless ferrite) have phase velocity in one direction and no mode propagates in the opposite direction.

We recall that this result applies to an indefinite structure and does not include all the modes with complex propagation constants which exist also for lossless ferrite, and are actually the only existing modes for lossy ferrite.

In order to verify these theoretical results we considered a small cross section guide with dimensions 10.25 by 5.45 mm loaded with a ferrite slab 1.5 mm thick against the side wall. The ferrite was transversely magnetized by an applied dc magnetic field ranging in value up to 4400 Oe. The microwave field was investigated by plunging a small dipole in the guide and displacing it in all coordinate directions, making use of some different mechanical arrangements (described in Bujatti [7]). Different ferrite slabs and different lengths of the slab were used but no appreciable modification was found in the field pattern. Also, more accurate polishing of the ferrite slab did not affect the measurements, i.e., the effects of accidental imperfections on the surface of the slab were negligible. Finally, the field pattern is not affected by the termination, for a ferrite slab sufficiently long, for any value of the applied magnetic field. Once this was established the guide was left most of time without any termination and the probe introduced from the open end. The measurements were repeated for different values of the applied magnetic field with a fixed working frequency equal to 9700 MHz and the following results were established:

1) No output can be detected at the end of the structure except when the applied magnetic field H assumes a value $H'<H<H''$ (with $H'=1500$ Oe and $H''=2800$ Oe for R4 Ferramic and a working frequency equal to 9700 MHz as previously stated) if the ferrite slab is sufficiently long (more than 3–4 cm).

2) For applied magnetic field values $H'<H<H''$ the microwave field is exponentially decreasing in the x direction at a rate varying with the applied magnetic field, reaching a maximum of about 8 dB/mm in the center of the range $H'-H''$, and a value of about 3.5 dB/mm at the boundaries of the range $H'-H''$. The microwave field is a surface wave guided by the ferrite [Fig. 2(b)]. By rotating the probe around the x axis, from the z to the y direction the presence of a y component of the microwave field was found, which should not be there if only the zero-order mode was excited.

For applied field values $H'<H<H''$, the dependence on y always shows oscillations of the microwave field along the length of the guide (Figs. 4, 5, and 6) which disappear for applied magnetic fields out of the range $H'-H''$ (Fig. 3. Figures 4 and 5 show how the pattern changes by moving the probe along the ferrite slab (z direction) and away from the ferrite slab (x direction).

Finally, always for applied field values $H'<H<H''$, the dependence on z is shown in Fig. 6 and again oscillations are detected showing the presence of higher order modes. A tentative modal analysis mainly suggests the presence of the second- and four-order harmonics.

3) For applied magnetic fields $H<H'$ (including all negative values) and $H>H''$ (up to the highest value tested equal to 4400 Oe) the dependence of the microwave field on x, normally to the ferrite slab, was found to be sinusoidal; the dependence on y, along the length of the guide, was found exponentially decreasing and the field was found to be constant with z. The overall configuration is the same expected for a TE_{10} in a guide under cutoff loaded with a slab of dielectric material having relative dielectric constant equal to 11 and scalar permeability varying around one depending on the applied magnetic field. The attenuation in the y direction depends on the value of the applied magnetic field as expected by the

[1] See Lax and Button [5], p 395.